Throughout: Over $\mathbb{C}$

Goals: ...

Basics of linear series

X a scheme, L a l.b. on X

The complete linear series corr. to L is

$$|L| := \mathbb{P}(H^0(X,L))$$

(If D is a divisor, $|D| = |\mathcal{O}(D)|$)

In general:

$V \subseteq H^0(X,L)$ a subspace, then $|V| = \mathbb{P}(V)$ a linear series.

We have an evaluation map

$ev: V \otimes \mathcal{O}_X \longrightarrow L$ given by evaluating sections

Fiber-wise: over x, $V \longrightarrow \overbrace{H^0(L|_x)}^{\text{one-dim}}$

so ev determines a map to $\mathbb{P}(V)$ whenever these are all surjective

quotients (i.e. ev is surj.)

Taking cokernel...

$$V \otimes \mathcal{O}_X \longrightarrow L \longrightarrow B \to 0$$

The base locus of $|V|$, $Bs(|V|)$ is the scheme-theoretic support of B (i.e. points at which all sections vanish)

Tensor by L^* ...

$$V \otimes L^* \to \mathcal{O}_X \to B \otimes L^* \to 0$$

The <u>base ideal</u> of V is image of $V \otimes L^* \to \mathcal{O}_X$.

$|V|$ is <u>base-point free</u> if $Bs(|V|) = \emptyset$

$\iff V \otimes \mathcal{O}_X \to L$ is surjective

$\iff$ ev determines a morphism $\phi_{|V|} : X \to \mathbb{P}(V)$

In general (for $V \neq 0$), $|V|$ determines a rational map

$$\phi_{|V|} : X \dashrightarrow \mathbb{P}(V)$$

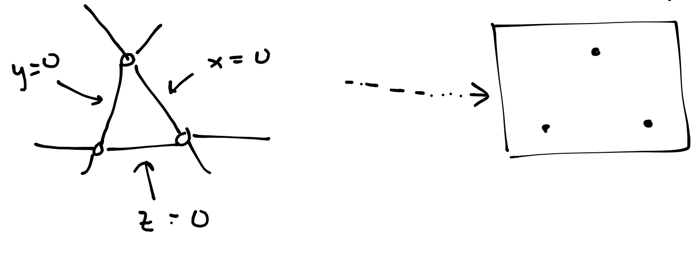
defined on $X - Bs(|V|)$.

<u>Ex</u>: Cremona transf.

Consider $L = \mathcal{O}_{\mathbb{P}^2}(2)$, $V = (xy, xz, yz)$

$$\phi_{|V|} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

All sections are $0 \iff x = y = 0$, $x = z = 0$, or $y = z = 0$



Projection:

If $V \subseteq W \subseteq H^0(L)$, then

$$\begin{tikzcd} W \arrow[r, two heads] & Q \\ V \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

$\leftarrow$ 1·dim

$\leftarrow$ 0 if $V \subseteq \ker(W \to Q)$

So $\pi : \mathbb{P}(W) \dashrightarrow \mathbb{P}(V)$ is proj. away from $\mathbb{P}(W/V)$

so

$$\begin{tikzcd} X \arrow[r, dashed, "\phi_{|W|}"] \arrow[dr, dashed, "\phi_{|V|}"'] & \mathbb{P}(W) \arrow[d, dashed, "\pi"] \\ & \mathbb{P}(V) \end{tikzcd}$$

(X factors through projection ... base locus?...)

Amplitude

X complete (projective) scheme

L l.b. on X

L is <u>very ample</u> if any of the equivalent hold:

1.) $\phi_{|L|}$ is an embedding

2.) $\exists$ an embedding $X \to \mathbb{P}^n$ s.t. $L = \mathcal{O}_X(1)$

3.) Any length 2 subscheme $Z \subseteq X$ imposes independent conditions on sections of L, i.e.

$$H^0(L) \to H^0(L|_Z) \text{ is surj. } \forall Z.$$ (Generalizations?)

L is <u>ample</u> if $L^{\otimes m}$ is very ample for some $m > 0$.

Story for curves:

Thm: C a curve of genus g.

a.) If $g \geq 2$, then K_C is globally generated, and $|mK_C|$ is very ample when $m \geq 3$.

b.) L a l.b. on C, $\deg(L) = d$. If $d \geq 2g$, L is b.p.f., and if $d \geq 2g+1$ then L is very ample.